Separable simplicial algebras

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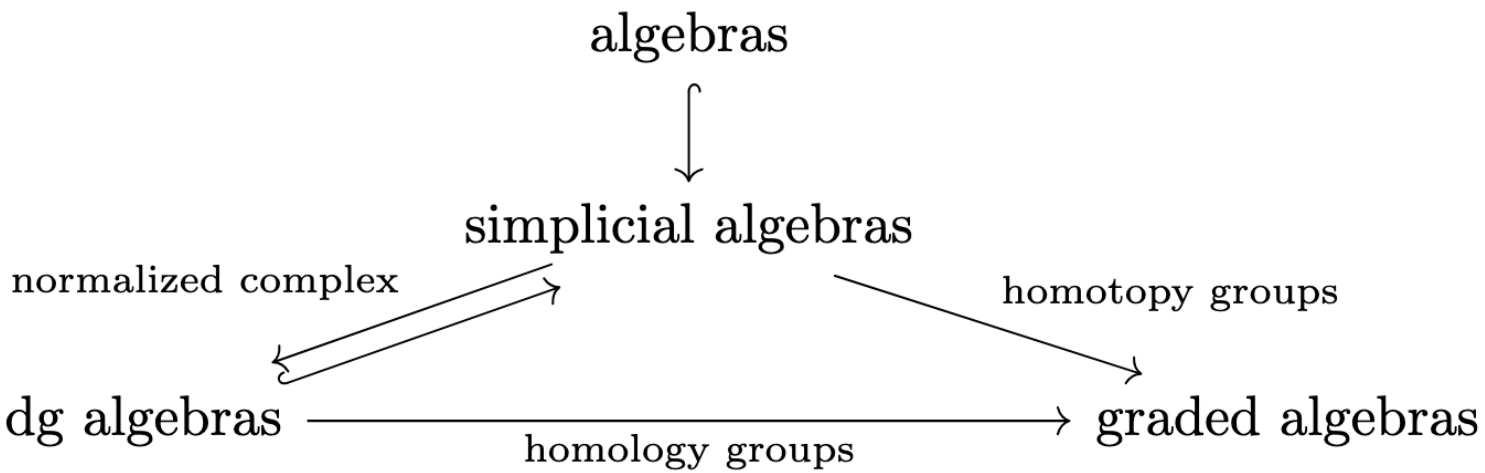
Why separable simplicial?

- Twisted Calabi-Yau algebras of dimension 0 $\iff$ separable (Reyes-Rogalski 22').
- Twisted Calabi-Yau algebras are studied in more general settings: differential graded (Ginzburg 07' etc.) $\to$connected graded (Reyes-Rogalski-J.Zhang 14' etc.) $\to$ group-graded (Baki 25' etc.) $\to$ monoidal category (Baki-Reyes 26').
- Separable algebras have been studied in (braided) monoidal categories (Pareigis 76', Van Oystaeyen-Y.Zhang 98') $\to$ (tensor-)triangulated categories (Balmer 11', 16')$\to$ symmetric monoidal ∞-categories (Naumann-Pol-Ramzi 26').

Why separable simplicial?

- Simplicial algebras are concrete algebraic objects to study, carry homotopic data simultaneously, and are naturally related to graded algebras and differential graded algebras.



Main questions

By an algebra or a ring, we always mean an associative unital one.

1. Define and characterize separable simplicial R-algebras.
2. Study related invariants such as the Brauer group.
3. Give a tentative definition of twisted Calabi-Yau simplicial algebras and investigate if twisted Calabi-Yau simplicial algebras of dimension 0 are precisely the separable ones.

Simplicial objects

Definition

A simplicial object X in a category $\mathcal{C}$ is a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$.

Equivalently, X consists of objects $\{X_n\}_{n\geq 0}$ with face maps $d_i : X_n \to X_{n-1}$ for $n \geq 1$ and degeneracy maps $s_i : X_n \to X_{n+1}$ for $n \geq 0$, where $0 \leq i \leq n$ satisfying the simplicial identities.

$$\cdots X_2 \;\begin{array}{c} \xrightarrow{\quad d_0 \quad} \\ \xleftarrow{\quad s_0 \quad} \\ \xrightarrow{\quad d_1 \quad} \\ \xleftarrow{\quad s_1 \quad} \\ \xrightarrow[\quad d_2 \quad]{} \end{array}\; X_1 \;\begin{array}{c} \xrightarrow{\quad d_0 \quad} \\ \xleftarrow{\quad s_0 \quad} \\ \xrightarrow[\quad d_1 \quad]{} \end{array}\; X_0$$

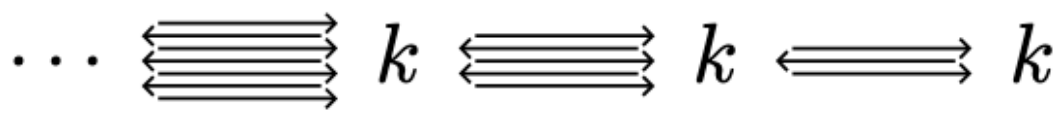
Examples of simplicial objects

Example

The representable functor $\Delta^n = \mathrm{Hom}_\Delta(-, [n])$ is a simplicial set and $\Bbbk\Delta^n$ is the simplicial $\Bbbk$-module freely generated by Δ^n.

Example

A commutative ring $\Bbbk$ can be naturally identified as the constant simplicial ring $\Bbbk\Delta^0$.

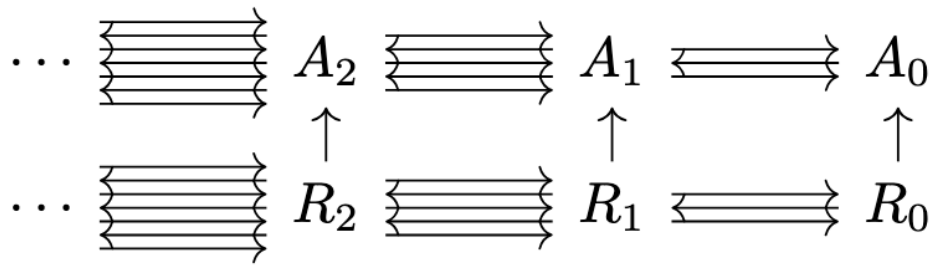


Example

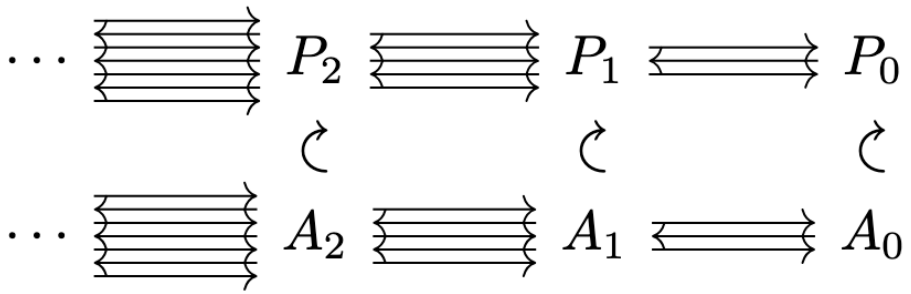
Singular and Hochschild chain complexes are simplicial abelian groups.

Simplicial algebras and simplicial modules

- A simplicial ring R is a simplicial object in Ring. Equivalently, it is a monoid in the category of simplicial abelian groups.
- A simplicial R-algebra A is a simplicial ring morphism $\eta : R \to A$ where $\eta(R) \subseteq Z(A)$ (Schwede-Shipley 00').



- Simplicial modules are defined as module objects over monoid and the tensor product is defined degreewise.



Simplicial enrichment

Let A be a simplicial R-algebra.

Definition (Quillen 67')

Let M and N be left simplicial A-modules. The simplicial hom-set $\mathbf{Hom}_A(M,N)$ is a simplicial R-module where

$$\mathbf{Hom}_A(M,N)_n = \mathrm{Hom}_{A\,\mathrm{sMod}}(R\Delta^n \otimes_R M, N).$$

Proposition (Hovey 99')

Let A be a simplicial R-algebra. Then $A\,\mathrm{sMod}$ is $R\,\mathrm{sMod}$-enriched.

Remark

Simplicial R-algebras A and B are simplicial Morita equivalent if $A\,\mathrm{sMod} \simeq B\,\mathrm{sMod}$ as enriched categories.

Projectivity in simplicial setting

Definition (HSZZ 26')

Let P be a left simplicial A-module. We say P is

(1) projective if $\mathrm{Hom}_A(P, -)$ is exact.

(2) simplicially projective if $\mathbf{Hom}_A(P, -)$ is exact.

(3) finitely generated if there exist finite $x_i \in P_{d_i}$ such that the smallest simplicial A-submodule of P containing $\{x_i\}$ is P.

(4) f.g. projective if it is finitely generated and projective.

(5) basic projective if P is a direct summand of some $A^{\oplus n}$.

Remark

Basic projectives fit better than f.g. projectives in simplicial Morita theory.

Separability in simplicial setting

Definition (HSZZ 26')

Let A be a simplicial R-algebra and $A^e = A \otimes_R A^{op}$. We say A is

(1) separable over R if the left simplicial A^e-module morphism

$$A^e \to A \qquad a \otimes b \mapsto ab$$

splits. That is, A is a projective left simplicial A^e-module.

(2) Azumaya over R if A is central and separable over R.

Remark

One can also define A to be Azumaya if A^e is simplicial Morita equivalent to R (i.e, $A^e\,\mathrm{sMod} \simeq R\,\mathrm{sMod}$ as enriched categories.)

Auslander-Goldman Theorem

Theorem (HSZZ 26')

Let A be a simplicial R-algebra. The following are equivalent.

(1) *A is Azumaya (central separable) over R.*

(2) *A is an A^e-progenerator and A is central over R.*

(3) *A is an R-progenerator and the canonical morphism $A^e \to \mathbf{Hom}_R(A, A)$ given by left A^e-actions is an isomorphism of simplicial R-algebras.*

Remark

It is not known in general whether central-separable Azumaya monoids agree with Morita-theoretic Azumaya monoids (Pareigis 76', Borceux-Vitale 02'). Our simplicial Auslander-Goldman theorem answers this question for simplicial algebras.

Some rigidity results

Proposition (HSZZ 26')

Let A be a simplicial $\Bbbk$-algebra. Then A is separable over $\Bbbk$ if and only if A_0 is separable over $\Bbbk$ and $A \cong A_0$, the constant simplicial algebra determined by A_0.

Theorem (HSZZ 26')

Let A be a simplicial R-algebra. Then A is separable over R if and only if $Z(A)$ is separable over R, A_0 is Azumaya over $Z(A)_0$, and $A \cong Z(A) \otimes_{Z(A)_0} A_0$ as simplicial $Z(A)$-algebras.

Theorem (HSZZ 26')

Let R be a commutative simplicial ring. Then $Br(R) \cong Br(R_0)$.

Twisted Calabi-Yau conditions

Definition (Tentative.)

Let A be a simplicial R-algebra. We say A is twisted Calabi-Yau of dimension d if

(1) there is a resolution of length d

$$0 \to P_d \to \cdots \to P_1 \to P_0 \to A \to 0$$

by basic projective left simplicial A^e-modules

(2)

$$\mathbf{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0 & i \neq d \\ A^\mu & i = d \end{cases}$$

as right simplicial A^e-modules for $\mathbf{Ext}^i = H^i(R\mathbf{Hom})$ and some automorphism $\mu : A \to A$.

Example

Example

Let R be a commutative simplicial $\mathbb{k}$-algebra.

Let $A = R[x]$ where $A_n = R_n[x]$.

Here A can be viewed as a functor

$$A : \Delta^{op} \xrightarrow{R} \mathbb{k}\,\mathrm{Alg} \xrightarrow{(-)[x]} \mathbb{k}\,\mathrm{Alg}\,.$$

There is a resolution of left simplicial A^e-modules

$$0 \to A^e = R[x,y] \xrightarrow{x-y} A^e = R[x,y] \xrightarrow{mult.} A = R[x] \to 0.$$

$A = R[x]$ is a twisted Calabi-Yau simplicial algebra of dimension 1.

Some rigidity results

Lemma

Let P be a basic projective left simplicial A-module. Then

$$A \otimes_{A_0} P_0 \cong P \qquad a \otimes p \mapsto a \cdot s_0^n(p)$$

is an isomorphism of left simplicial A-modules.

Corollary

Suppose there is a basic projective resolution $Q_1 \to Q_0 \to M \to 0$ of left simplicial A-modules. Then $A \otimes_{A_0} M_0 \cong M$ as left simplicial A-modules.

Proposition

Let A be a twisted Calabi-Yau simplicial R-algebra. Then

(1) $A \cong A^e \otimes_{A_0^e} A_0$ *as left simplicial A^e-modules.*

(2) A_0 *is an ordinary twisted Calabi-Yau R_0-algebra.*

(3) *If $R = \Bbbk$ is the constant simplicial ring, then $A \cong A_0$ which is the constant simplicial algebra determined by A_0.*

Remark

Over $\Bbbk$, the twisted Calabi-Yau simplicial algebras are no different from ordinary twisted Calabi-Yau algebras. This is consistent with our rigidity result for separable simplicial $\Bbbk$-algebras.

Questions for future studies

(1) Twisted Calabi-Yau of dim 0 vs. separable.

– If A is a twisted Calabi-Yau simplicial R-algebra of dimension 0, it is (basic) projective as a left simplicial A^e-module, whence separable.

– If A is separable, then $\mathbf{Ext}^i_{A^e}(A, A^e) = 0$ for $i > 0$. It remains to show $\mathbf{Hom}_{A^e}(A, A^e) \cong A$.

(2) Nakayama automorphism.

– A twisted Calabi-Yau simplicial R-algebra A satisfies $A \cong A^e \otimes_{A_0^e} A_0$. Is the Nakayama automorphism of A always extended from that of A_0?

(3) Basic projective vs. f.g. projective.

– The rigidity of twisted Calabi-Yau simplicial algebras comes from the basic projective resolution in the definition. What changes if the resolution uses arbitrary f.g. projectives?

Thank you!

References I

1. Y. S. Baki, *Group-graded twisted Calabi–Yau algebras*, J. Pure Appl. Algebra **229** (2025), no. 1, 107849.
2. P. Balmer, *Separability and triangulated categories*, Adv. Math. **226** (2011), no. 5, 4352–4372.
3. P. Balmer, *Separable extensions in tensor-triangular geometry and generalized Quillen stratification*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 4, 907–925.
4. F. Borceux and E. M. Vitale, *Azumaya categories*, Appl. Categ. Structures **10** (2002), no. 5, 449–467.
5. V. Ginzburg, *Calabi–Yau algebras*, arXiv:math/0612139v3, 2007.

References II

6 X. Huang, G. Sun, E. Zhang, and J. J. Zhang, *Introduction to separable simplicial algebras*, arXiv:2610.38229, 2026.

7 Mark Hovey, *Model Categories*, Mathematical Surveys and Monographs, vol.63, American Mathematical Society, Providence, RI, 1999.

8 N. Naumann, L. Pol, and M. Ramzi, *Separable commutative algebras in equivariant homotopy theory*, J. Reine Angew. Math. **837** (2026), 55–92.

9 B. Pareigis, *Non-additive ring and module theory IV. The Brauer group of a symmetric monoidal category*, in *Brauer Groups* (D. Zelinsky, ed.), Lecture Notes in Math., vol. 549, Springer, 1976, pp. 112–133.

References III

10 M. L. Reyes and D. Rogalski, *Graded twisted Calabi-Yau algebras are generalized Artin-Schelter regular*, *Nagoya Math. J.* **245** *(2022), 100–153.*

11 *M. Reyes, D. Rogalski, and J. J. Zhang, Skew Calabi–Yau algebras and homological identities, Adv. Math.* **264** *(2014), 308–354.*

12 *S. Schwede and B. E. Shipley, Algebras and modules in monoidal model categories, Proc. London Math. Soc. (3)* **80** *(2000), no.2, 491–511.*

13 *F. Van Oystaeyen and Y. Zhang, The Brauer group of a braided monoidal category, J. Algebra* **202** *(1998), no. 1, 96–128.*